NONSTATIONARY HEATING OF A CONICAL ROD

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The Galerkin-Kantorovich method is used to obtain an approximate solution of the nonstationary problem of heat conduction for a thin conical rod.

In a number of branches of technology (motor-building, aviation, etc.) finned surfaces are often used to promote cooling efficiency. Conical fins are most commonly employed. Considerable interest attaches to problems of the heating of thin fins. In designing finned surfaces it is first necessary to determine the temperature distribution over the fin. In a number of studies [1–3] the solution of the stationary problem of the heating of a thin conical rod is given in terms of Bessel functions. It is also important to obtain a solution of the nonstationary problem.

We will consider the heating of a thin rod, i.e., an element with high thermal conductivity whose transverse dimensions are small compared with its length. Then we can neglect transverse temperature gradients and consider the one-dimensional formulation of the problem. We can estimate the error due to the assumption of one-dimensionality, for example, from the solution of the one-dimensional problem of heat conduction for an infinite plate [4], considering the heat conduction over the thickness of the rod under the least favorable heating conditions.

Mathematically the problem is formulated as follows (Fig. 1):

$$\frac{\partial t}{\partial \tau} = a\,\frac{\partial^2 t}{\partial x^2} - \frac{2a}{H-x}\,\frac{\partial t}{\partial x} - \frac{2\alpha}{c\gamma (H-x)\sin\beta}\,(t-t_m), \quad (1)$$

$$-\lambda\,\frac{\partial t(0,\ \tau)}{\partial x} = \alpha_1\,[t_{m_1} - t(0,\ \tau)],\quad \frac{\partial t(h,\ \tau)}{\partial x} = 0, \quad (2)$$

$$t(x,\ 0) = f(x). \quad (3)$$

We note that in the case of a conical noncircular rod the mathematical formulation of the problem remains the same (correct to the constant coefficients in Eq. (1)).

In the limit as $\beta \to 0$ (h and r remain constant) Eq. (1) goes over into the heat conduction equation for a cylindrical rod

$$\frac{\partial t_c}{\partial \tau} = a\,\frac{\partial^2 t_c}{\partial x^2} - \frac{2\alpha}{c\gamma r}\,(t_c - t_m). \quad (4)$$

The solution of the problem (1)–(3) will be found in the form

$$t(x,\ \tau) = t_{st}(x) + u(x,\ \tau). \quad (5)$$

The stationary solution $t_{st}(x)$ is found from (1) with boundary conditions (2). Then $u(x, \tau)$ is determined from the problem

$$L[u] = \frac{\partial u}{\partial \tau} - a\,\frac{\partial^2 u}{\partial x^2} + \frac{2a}{H-x}\,\frac{\partial u}{\partial x} + \frac{2\alpha}{c\gamma (H-x)\sin\beta}\,u = 0, \quad (6)$$

$$\lambda\,\frac{\partial u(0,\ \tau)}{\partial x} = \alpha_1 u(0,\ \tau),\quad \frac{\partial u(h,\ \tau)}{\partial x} = 0, \quad (7)$$

$$u(x,\ 0) = f(x) - t_{st}(x) = f_1(x). \quad (8)$$

To determine $t_{st}(x)$ we employ the Galerkin method [5], using as coordinate function the known stationary solution of problems (4) and (2), which has the form

$$t_{st.c} = t_m + \frac{\alpha_1 (t_{m_1} - t_m)}{\alpha_1 \operatorname{ch}\sqrt{2\alpha/\lambda r}\,h + \lambda\sqrt{2\alpha/\lambda r}\operatorname{sh}\sqrt{2\alpha/\lambda r}\,h} \times \operatorname{ch}\sqrt{\frac{2\alpha}{\lambda r}}\,(x-h). \quad (9)$$

For the stationary case in the new variable $v(x)$ obtained from $t_{st}(x)$ by means of the relation $t_{st}(x)\ t_{m_1} - v(x)$, Eq. (1) is written as follows:

$$M[v] = (H-x)\,\frac{d^2 v}{dx^2} - 2\,\frac{dv}{dx} - \frac{2\alpha}{\lambda\sin\beta}\,v + \frac{2\alpha\,(t_{m_1} - t_m)}{\lambda\sin\beta} = 0. \quad (10)$$

We will find $v(x)$ in the form

$$v(x) = A\,[t_{m_1} - t_{st.c}(x)],$$

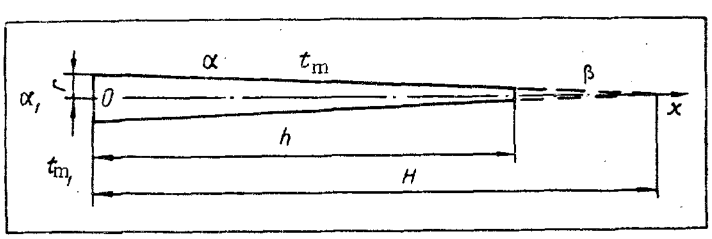

Fig. 1. Conical rod.

where the constant A is found from the Galerkin equation

$$\int_0^h M[v(x)]\,[t_{m_1} - t_{st.c}(x)]\,dx = 0.$$

The stationary solution in dimensionless form is

$$\Theta_{st} = A\left[1 - \frac{\mathrm{Bi}_1\,\mathrm{ch}[\bar{\varkappa}(1-\bar{x})]}{\mathrm{Bi}_1\,\mathrm{ch}\,\bar{\varkappa} + \varkappa\,\mathrm{sh}\,\bar{\varkappa}}\right], \qquad (11)$$

where

$$A = \frac{2\mathrm{Bi}}{\sin\beta}\left[-1 + \frac{1}{\bar{\varkappa}(\mathrm{cth}\,\bar{\varkappa} + \bar{\varkappa}/\mathrm{Bi}_1)}\right]\times$$

$$\times\left\{-\frac{\bar{\varkappa}\bar{H} + \mathrm{th}\,\dfrac{\bar{\varkappa}}{2}}{\mathrm{cth}\,\bar{\varkappa} + \bar{\varkappa}/\mathrm{Bi}_1} + \frac{2\mathrm{Bi}}{\sin\beta}\left[-1 + \frac{2}{\bar{\varkappa}(\mathrm{cth}\,\bar{\varkappa} + \bar{\varkappa}/\mathrm{Bi}_1}\right] + \right.$$

$$\left. + \frac{-\dfrac{\bar{\varkappa}^2}{4\mathrm{sh}^2\bar{\varkappa}} + \dfrac{3}{4} - \left(\dfrac{1}{\mathrm{sh}^2\bar{\varkappa}} + \dfrac{\mathrm{cth}\,\bar{\varkappa}}{\bar{\varkappa}}\right)\mathrm{Bi}\,\mathrm{tg}\,\dfrac{\beta}{2}}{(\mathrm{cth}\,\bar{\varkappa} + \bar{\varkappa}/\mathrm{Bi})^2}\right\}^{-1}.$$

Figure 2 presents curves of the distribution of stationary temperatures along the rod calculated from the formulas we have obtained (approximate solution) and by the method of pivotal condensation (exact solution). The agreement of the results is good. The results of calculations based on a series of other values of the parameters also indicate that using only the first approximation of the Galerkin method ensures good accuracy.

We will seek the approximate solution of problem (6)–(8) in accordance with the Galerkin-Kantorovich method [5] in the form

$$u^{(n)}(x, \tau) = \sum_{k=1}^{n} \cos\mu_k(1 - x/h)\,\varphi_k(\tau), \qquad (12)$$

where μ_k is found from the equation $\mathrm{tg}\,\mu_k = \frac{\mathrm{Bi}_1}{\mu_k}$, $\varphi_k(\tau)$ are determined from the system of Galerkin equations

$$\int_0^h L\left[\sum_{k=1}^{n} \cos\mu_k(1 - x/h)\,\varphi_k(\tau)\right]\cos\mu_m(1 - x/h)\,dx = 0, \quad m = 1, \ldots, n.$$

Hence we obtain

$$\frac{d\bar{\varphi}_m}{dF_0} + a_m\bar{\varphi}_m + {\sum_{k=1}^{n}}' a_{mk}\bar{\varphi}_k = 0, \quad m = 1, \ldots, n, \qquad (13)$$

a prime on the summation sign indicating that the term k = m is not included in the summation.

It is easy to see that $e_m > 0$, $d_m > 0$ and $a_m > \mu_m^2$.

Thus, the problem of determining $u(x, \tau)$ reduces to solving system (13) for φ_m. The integration of this system leads to algebraic operations and in the general eral case can be successfully performed on electronic computers. A desk calculator can be used for calculating first approximations.

In the limit when the conical rod becomes cylindrical, system (13) breaks down and is written in the following simple form:

$$\frac{d\bar{\varphi}_m}{dF_0} + \left(\mu_m^2 + \frac{2\mathrm{Bi}}{\bar{r}}\right)\bar{\varphi}_m = 0, \quad m = 1, \ldots, n. \qquad (14)$$

The roots of the characteristic equation of system (13) are negative or have negative real parts, which

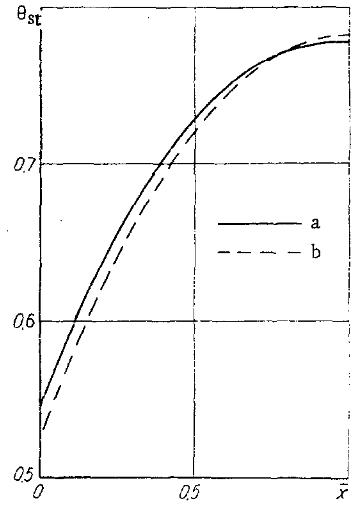

Fig. 2. Distribution of stationary temperature Θ_{st} along length of rod ($t(x, 0) = t_m$; $\mathrm{Bi}_1 = 1$; $\mathrm{Bi} = 0.2$; $\beta = 0.05$; $\bar{H} = 4$): a) approximate solution; b) exact solution.

corresponds to the physical significance of the heat conduction problem considered. For first approximations this result is easily obtained in general form if the Routh-Hurwitz criterion is employed.

To determine $\varphi_m(0)$ we substitute (12) into (8), multiply both sides of the equation obtained by $\cos\mu_m \cdot (1 - x/h)$ and integrate from 0 to h. We then obtain

$$\bar{\varphi}_m(0) = \frac{2\mu_m}{\mu_m + \sin\mu_m\cos\mu_m}\int_0^1 \frac{f(\bar{x}) - t_{st}(\bar{x})}{t_{m_1} - t_m}\cos\mu_m(1 - \bar{x})\,d\bar{x}.$$

If $f(x) = t_m$, it follows that

$$\bar{\varphi}_m(0) = \frac{2\mu_m^2\sin\mu_m}{\mu_m + \sin\mu_m\cos\mu_m}\left(\frac{A-1}{\mu_m^2} - \frac{A}{\mu_m^2 + \bar{\varkappa}^2}\right).$$

The final form of the solution of our problem is then

$$\Theta^{(n)} = \Theta_{st}(\bar{x}) - \sum_{k=1}^{n}\cos\mu_k(1 - \bar{x})\,\bar{\varphi}_k(F_0). \qquad (15)$$

In the limiting case of a cylindrical rod and with $f(x) = t_m$ we obtain the known formula [4].

Figure 3 presents curves of $\Theta^{(n)}$ as a function of the number F_0 for our example of the heating of a rod. The behavior of the curves indicates that the convergence

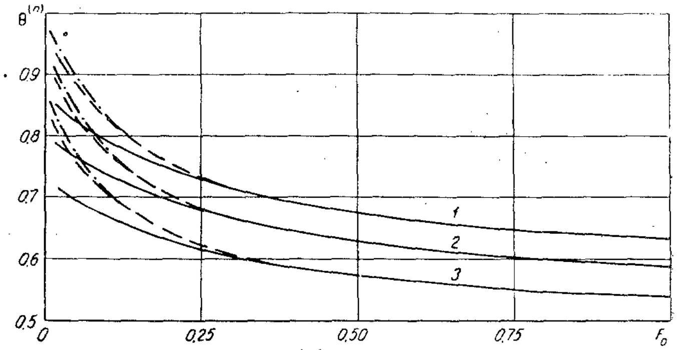

Fig. 3. Temperature $\Theta^{(n)}$ as a function of the number F_0($t(x, 0) = t_m$; $Bi_1 = 1$; $Bi = 0.2$; $\beta = 0.05$; $\overline{H} = 4$): solid line—for $\Theta^{(1)}$, broken line—$\Theta^{(2)}$, dot-dash line—$\Theta^{(3)}$; 1, 2, 3) at $x = 0.2$, 0.1, and 0, respectively.

of the successive approximations is good and that it is perfectly sufficient to confine oneself to the first three.

We also used the above method to solve the problem of the heating of a conical rod when a conductive heat flow is supplied through the smaller base of the cone, the larger base being thermally insulated. The method can also be employed to calculate the heating of a rod under other types of boundary conditions.

NOTATION

t—temperature in rod; x—coordinate; τ—time; λ—heat conductivity; c—specific heat; γ—specific weight; a—thermal diffusivity; H—height of complete cone; t_m and α—temperature of medium and heat transfer coefficient at lateral surface; t_{m1} and α_1—the same quantities at the larger base of the cone; β—cone half angle; h—height of truncated cone; r—radius of larger base; t_{st}—stationary temperature in rod; $t^{(n)}$—nonstationary temperature in n-th approximation; $Bi = \alpha h/\lambda$, $Bi_1 = \alpha_1 h/\lambda$, $\overline{H} = H/h$, $\overline{\varkappa} = h\sqrt{2\alpha/\lambda r}$, $\overline{x} = x/h$, $\overline{r} = r/h$; $\theta_{st} = (t_{m_1} - t_{st})/(t_{m_1} - t_m)$, $\theta^{(n)} = (t_{m_1} - t^{(n)})/(t_{m_1} - t_m)$, $\overline{\varphi}_m = \varphi_m/(t_{m_1} - t_m)$, $F_0 = a\tau/h^2$, $e_m = (2\overline{H} - 1)/2 + \overline{H}\,\dfrac{\sin 2\mu_m}{2\mu_m} - \dfrac{1}{2}\,\dfrac{\sin^2\mu_m}{\mu_m^2}$, $d_m = \mu_m^2 \times \left(e_m + 2\,\dfrac{\sin^2\mu_m}{\mu_m^2}\right) + \dfrac{2Bi}{\sin\beta}\left(1 + \dfrac{\sin 2\mu_m}{2\mu_m}\right)$, $f_{mk} = \dfrac{4\mu_k^2}{\mu_k^2 - \mu_m^2}\left(1 - \dfrac{\cos\mu_m}{\cos\mu_k}\right)$, $a_m = \dfrac{d_m}{e_m}$, $a_{mk} = \dfrac{f_{mk}}{e_m}$—dimensionless and criterial quantities.

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